A HARTOGS TYPE EXTENSION THEOREM FOR GENERALIZED (*N,k*)-CROSSES WITH PLURIPOLAR SINGULARITIES

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Abstract. The aim of this paper is to present an extension theorem for (N,k)-crosses with pluripolar singularities.

1. Introduction. Statement of the main result

1.1. **Introduction.** The topic of separately holomorphic functions have a long history in complex analysis. The problem was first investigated by W. F. Osgood in [Osg 1899]. Seven years later F. Hartogs in [Har 1906] proved his famous theorem stating that every separately holomorphic function is, in fact, holomorphic. Since then the interest switched to more general problem - whether a function f defined on a product of two domains $D\times G$, being separately holomorphic on some subsets $A\subset D$ and $B\subset G$ is holomorphic on the whole $D\times G$ (see for example papers of M. Hukuhara [Huk 1942] and T. Terada [Ter 1967]) - which led to the question of possible holomorphic extension of a function separately holomorphic on the objects called crosses.

In a recent paper [Lew 2012] A. Lewandowski introduces an object called generalized (N,k)-cross $\mathbf{T}_{N,k}$, being a generalization of (N,k)-cross defined by M. Jarnicki and P. Pflug in [JarPfl 2010], and proves an extension theorem for this new type of cross with analytic singularities. In this paper we will prove a similar extension theorem for $\mathbf{T}_{N,k}$ crosses with pluripolar singularities, being a generalization of Theorem 10.2.9 from [JarPfl 2011] and Main Theorem from [JarPfl 2003]. We will also introduce other type of generalized (N,k)-crosses called $\mathbf{Y}_{N,k}$ crosses, being more natural object to consider in light of Theorem 3.6. This theorem will turn out to be a very strong tool, allowing us to prove two Hartogs-type extension theorems for the functions separately holomorphic on $\mathbf{X}_{N,k}$, $\mathbf{T}_{N,k}$ and $\mathbf{Y}_{N,k}$ crosses, including the Main Theorem of this paper.

The paper is divided into four sections. In the first section we define generalized (N,k)-crosses and we state the Main Theorem. Section 2 contains some useful definitions and facts. Section 3 is dedicated to (N,k)-crosses - their properties and recent cross theorems. It also contains the statement of Theorem 3.6 and the proof of Main Theorem. In the last section we present the detailed proof of Theorem 3.6.

1.2. **Generalized (*N,k*)-crosses and the main result.** Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$ and let $A_j\subset D_j$ be locally pluriregular (see Definition 2.1), $j=1,\dots,N$, where $N\geq 2$. For $\alpha=(\alpha_1,\dots,\alpha_N)\in\{0,1\}^N$ and $B_j\subset D_j$, $j=1,\dots,N$, define:

$$\mathcal{X}_\alpha := \mathcal{X}_{1,\alpha_1}\times\dots\times\mathcal{X}_{N,\alpha_N},\quad \mathcal{X}_{j,\alpha_j} := \begin{cases} D_j & \text{when} \quad \alpha_j=1\\ A_j & \text{when} \quad \alpha_j=0\end{cases},\quad j=1,\dots,N,$$

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$$B_0^\alpha := \prod_{j\in\{1,\dots,N\}:\,\alpha_j=0} B_j, \quad B_1^\alpha := \prod_{j\in\{1,\dots,N\}:\,\alpha_j=1} B_j.$$

For $\alpha \in \{0,1\}^N$ we merge $c_0 \in D_0^\alpha$ and $c_1 \in D_1^\alpha$ into $(\overleftrightarrow{c_0, c_1}) \in \prod_{j=1}^n D_j$ by putting variables in right places.

We also use the following convention: for $D \subset D_0^\alpha$, $G \subset D_1^\alpha$, $\alpha \in \{0,1\}^N$, define

$$\overleftrightarrow{D \times G} := \{(\overleftrightarrow{a, b}) : \ a \in D,\ b \in G\}.$$

To simplify the notation let us define families

$$\mathcal{T}_k^N := \{\alpha \in \{0,1\}^N : \ |\alpha| = k\}, \quad \mathcal{Y}_k^N := \{\alpha \in \{0,1\}^N : \ 1 \le |\alpha| \le k\}.$$

Definition 1.1. For a $k \in \{1,\dots,N\}$ we define an (N,k)-*cross*

$$\mathbf{X}_{N,k} = \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N) := \bigcup_{\alpha\in\mathcal{T}_k^N} \mathcal{X}_\alpha.$$

For $\alpha \in \mathcal{Y}_k^N$ let $\Sigma_\alpha \subset A_0^\alpha$ and put

$$\mathcal{X}_\alpha^\Sigma := \{z \in \mathcal{X}_\alpha : \ z_\alpha \notin \Sigma_\alpha\}, \quad \alpha \in \mathcal{Y}_k^N,$$

where z_α denotes the projection of z on D_0^α.

Definition 1.2. We define a *generalized* (N,k)-*cross* $\mathbf{T}_{N,k}$

$$\mathbf{T}_{N,k} = \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha\in\mathcal{T}_k^N}) := \bigcup_{\alpha\in\mathcal{T}_k^N} \mathcal{X}_\alpha^\Sigma$$

and a *generalized* (N,k)-*cross* $\mathbf{Y}_{N,k}$

$$\mathbf{Y}_{N,k} = \mathbb{Y}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha\in\mathcal{Y}_k^N}) := \bigcup_{\alpha\in\mathcal{Y}_k^N} \mathcal{X}_\alpha^\Sigma.$$

Observe that always $\mathbf{T}_{N,k} \subset \mathbf{Y}_{N,k}$.

Example 1.3. To see the difference between generalized $\mathbf{T}_{N,k}$ and $\mathbf{Y}_{N,k}$ consider for example $N = 3$, $k = 2$, and let

$$\Sigma_{(1,1,0)} = \{z_3\} \subset A_3, \quad \Sigma_{(1,0,1)} = \{z_2\} \subset A_2, \quad \Sigma_{(0,1,1)} = \{z_1\} \subset A_1,$$
$$\Sigma_\alpha = \varnothing,\ \alpha \in \mathcal{Y}_2^3 \setminus \mathcal{T}_2^3.$$

Observe that if for all $\alpha \in \mathcal{Y}_k^N$ we have $\Sigma_\alpha = \varnothing$, then

$$\begin{aligned}\mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha\in\mathcal{T}_k^N}) &= \mathbb{Y}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha\in\mathcal{Y}_k^N}) \\ &= \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N).\end{aligned}$$

Moreover, for $k = 1$ we have $(\Sigma_\alpha)_{\alpha\in\mathcal{T}_k^N} = (\Sigma_\alpha)_{\alpha\in\mathcal{Y}_k^N} = (\Sigma_j)_{j=1}^N$ and we use the simplified notation

$$\mathbf{T}_{N,1} = \mathbf{Y}_{N,1} =: \mathbb{T}((A_j, D_j, \Sigma_j)_{j=1}^N).$$

Definition 1.4. For (N,k)-cross $\mathbf{W}_{N,k} \in \{\mathbf{X}_{N,k}, \mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}$ we define its *center* as

$$c(\mathbf{W}_{N,k}) := \mathbf{W}_{N,k} \cap (A_1 \times \ldots \times A_N).$$

Definition 1.5. For a cross $\mathbf{X}_{N,k} = \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N)$ we define its *hull*

$$\widehat{\mathbf{X}}_{N,k} = \widehat{\mathbb{X}}_{N,k}((A_j, D_j)_{j=1}^N) := \Big\{(z_1,\dots,z_N) \in D_1\times\ldots\times D_N : \ \sum_{j=1}^N \mathbf{h}_{A_j,D_j}(z_j) < k\Big\},$$

where $\mathbf{h}_{B,D}$ denotes relative extremal function of B with respect to D (see Definition 2.1).

Let $\mathbf{W}_{N,k} \in \{\mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}$ and let $M \subset \mathbf{W}_{N,k}$. For an $\alpha \in \mathcal{Y}_k^N$ and for an $a \in A_0^\alpha$ let $M_{a,\alpha}$ denote the fiber

$$M_{a,\alpha} := \{z \in D_1^\alpha : \ (\overleftrightarrow{a,z}) \in M\}.$$

For $(z', z'') \in \prod_{j=1}^{k} D_j \times \prod_{j=k+1}^{N} D_j$, $k \in \{1, \ldots, N-1\}$, define

$$M_{(z',\cdot)} := \{b \in \prod_{j=k+1}^{N} D_j : (z', b) \in M\}, \ M_{(\cdot,z'')} := \{a \in \prod_{j=1}^{k} D_j : (a, z'') \in M\}.$$

Definition 1.6. Let $M \subset \mathbf{T}_{N,k}$ be such that for all $\alpha \in \mathcal{T}_k^N$ and for all $a \in A_0^\alpha \setminus \Sigma_\alpha$ the set $D_1^\alpha \setminus M_{a,\alpha}$ is open. A function $f : \mathbf{T}_{N,k} \setminus M \to \mathbb{C}$ is called *separately holomorphic on* $\mathbf{T}_{N,k} \setminus M$ $(f \in \mathcal{O}_\mathcal{S}(\mathbf{T}_{N,k} \setminus M))$, if for all $\alpha \in \mathcal{T}_k^N$ and for all $a \in A_0^\alpha \setminus \Sigma_\alpha$, the function

$$(\dagger) \qquad D_1^\alpha \setminus M_{a,\alpha} \ni z \mapsto f((\overleftrightarrow{a,z})) =: f_{a,\alpha}(z)$$

is holomorphic.

For generalized (N,k)-cross $\mathbf{Y}_{N,k}$ we state an analogical definition.

Definition 1.7. Let $M \subset \mathbf{Y}_{N,k}$ be such that for all $\alpha \in \mathcal{Y}_k^N$ and for all $a \in A_0^\alpha \setminus \Sigma_\alpha$ the set $D_1^\alpha \setminus M_{a,\alpha}$ is open. A function $f : \mathbf{Y}_{N,k} \setminus M \to \mathbb{C}$ is called *separately holomorphic on* $\mathbf{Y}_{N,k} \setminus M$ $(f \in \mathcal{O}_\mathcal{S}(\mathbf{Y}_{N,k} \setminus M))$, if for all $\alpha \in \mathcal{T}_k^N$ and for all $a \in A_0^\alpha \setminus \Sigma_\alpha$, the function $(\dagger)$ is holomorphic.

Remark 1.8. Observe that if $f \in \mathcal{O}_\mathcal{S}(\mathbf{Y}_{N,k} \setminus M)$, then also for all $\alpha \in \mathcal{Y}_k^N$ and for all $a \in A_0^\alpha \setminus \Sigma_\alpha$ the function $(\dagger)$ is holomorphic.

Let $M \subset \mathbf{T}_{N,k}$. For $\alpha \in \mathcal{Y}_k^N$ and for $b \in D_1^\alpha$ let $M_{b,\alpha}$ denote the fiber

$$M_{b,\alpha} := \{z \in A_0^\alpha : \ (\overleftrightarrow{z,b}) \in M\}.$$

The following class of functions plays important role in the Main Theorem. It is a natural extension of class $\mathcal{O}_\mathcal{S}(\mathbf{T}_{N,k} \setminus M) \cap \mathcal{C}(\mathbf{T}_{N,k} \setminus M)$.

Definition 1.9. Let $M \subset \mathbf{T}_{N,k}$ be such that for all $\alpha \in \mathcal{T}_k^N$ and for all $a \in A_0^\alpha \setminus \Sigma_\alpha$ the set $D_1^\alpha \setminus M_{a,\alpha}$ is open. By $\mathcal{O}_\mathcal{S}^c(\mathbf{T}_{N,k} \setminus M)$ we denote the space of all functions $f \in \mathcal{O}_\mathcal{S}(\mathbf{T}_{N,k} \setminus M)$ such that for all $\alpha \in \mathcal{T}_k^N$ and for all $b \in D_1^\alpha$, the function

$$A_0^\alpha \setminus (\Sigma_\alpha \cup M_{b,\alpha}) \ni z \mapsto f((\overleftrightarrow{z,b})) =: f_{b,\alpha}(z)$$

is continuous.

The following theorem is the main result of this paper. It is an analogue and natural generalization of Theorem 10.2.9 from [JarPfl 2011]. It also extends the main result from [Lew 2012].

Main Theorem (Extension theorem for (N,k)-crosses with pluripolar singularities)**.** *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ be locally pluriregular, $j = 1, \ldots, N$. For $\alpha \in \mathcal{T}_k^N$ let $\Sigma_\alpha \subset A_0^\alpha$ be pluripolar. Let*

$$\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N), \quad \mathbf{T}_{N,k} := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha \in \mathcal{T}_k^N}).$$

Let M be a relatively closed, pluripolar subset of $\mathbf{T}_{N,k}$ such that for all $\alpha \in \mathcal{T}_k^N$ and all $a \in A_0^\alpha \setminus \Sigma_\alpha$ the fiber $M_{a,\alpha}$ is pluripolar. Let

$$\mathcal{F} := \begin{cases} \mathcal{O}_\mathcal{S}(\mathbf{X}_{N,k} \setminus M), & \textit{if for any } \alpha \in \mathcal{T}_k^N \textit{ we have } \Sigma_\alpha = \varnothing \\ \mathcal{O}_\mathcal{S}^c(\mathbf{T}_{N,k} \setminus M), & \textit{otherwise} \end{cases}.$$

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}_{N,k}$ *and a generalized* (N,k)*-cross* $\mathbf{T}'_{N,k} := \mathbb{T}_{N,k}((A_j,D_j)_{j=1}^N, (\Sigma'_\alpha)_{\alpha\in\mathcal{T}_k^N}) \subset \mathbf{T}_{N,k}$ *with* $\Sigma_\alpha \subset \Sigma'_\alpha \subset A_0^\alpha$, Σ'_α *pluripolar,* $\alpha \in \mathcal{T}_k^N$*, such that:*

- $\widehat{M} \cap (c(\mathbf{T}_{N,k}) \cup \mathbf{T}'_{N,k}) \subset M$,
- *for any* $f \in \mathcal{F}$ *there exists a function* $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{M})$ *such that* $\widehat{f} = f$ *on* $(c(\mathbf{T}_{N,k}) \cup \mathbf{T}'_{N,k}) \setminus M$,
- $\widehat{M}$ *is singular with respect to* $\{\widehat{f} :\ f \in \mathcal{F}\}$[(1)],
- *if* $M = \varnothing$*, then* $\widehat{M} = \varnothing$,
- *if for all* $\alpha \in \mathcal{T}_k^N$ *and all* $a \in A_0^\alpha \setminus \Sigma_\alpha$ *the fiber* $M_{a,\alpha}$ *is thin in* D_1^α*, then* $\widehat{M}$ *is analytic in* $\widehat{\mathbf{X}}_{N,k}$.

The following remark shows that Main Theorem can be stated analogously to Theorem 10.2.9 from [JarPfl 2011].

Remark 1.10. Observe that for any relatively closed pluripolar set $M \subset \mathbf{T}_{N,k}$ and for all $\alpha \in \mathcal{T}_k^N$ there exists a pluripolar set $\Sigma_\alpha^0 \subset A_0^\alpha$ such that $\Sigma_\alpha \subset \Sigma_\alpha^0$ and for all $a \in A_0^\alpha \setminus \Sigma_\alpha^0$ the fiber $M_{a,\alpha}$ is pluripolar. Then from Main Theorem we get the conclusion with $(\Sigma_\alpha)_{\alpha\in\mathcal{T}_k^N}$ and $\mathbf{T}_{N,k}$ substituted with $(\Sigma_\alpha^0)_{\alpha\in\mathcal{T}_k^N}$ and $\mathbf{T}_{N,k}^0 := \mathbb{T}_{N,k}((A_j,D_j)_{j=1}^N, (\Sigma_\alpha^0)_{\alpha\in\mathcal{T}_k^N}$.

2. Preliminaries

2.1. **Relative extremal function.**

Definition 2.1 (Relative extremal function)**.** Let D be a Riemann domain over $\mathbb{C}^n$ and let $A \subset D$. The *relative extremal function of A with respect to D* is a function

$$\mathbf{h}_{A,D} := \sup\{u \in \mathcal{PSH}(D) :\ u \le 1,\ u|_A \le 0\}.$$

For an open set $G \subset D$ we define $\mathbf{h}_{A,G} := \mathbf{h}_{A\cap G,G}$.

A set $A \subset D$ is called *pluriregular at a point* $a \in \overline{A}$ if $\mathbf{h}^*_{A,U}(a) = 0$ for any open neighborhood U of the point a, where $\mathbf{h}^*_{A,U}$ denotes the upper semicontinuous regularization of $\mathbf{h}_{A,U}$.

We call A *locally pluriregular* if $A \ne \varnothing$ and A is pluriregular at every point $a \in A$.

2.2. N**-fold crosses.**

Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$ and let $A_j \subset D_j$ be a nonempty set, $j = 1,\dots,N$, where $N \ge 2$. For $k = 1$, for historical reasons, we call $\mathbb{X}_{N,1}((A_j,D_j)_{j=1}^N)$ an *N-fold cross* $\mathbf{X}$ and we use the following notation

$$\mathbf{X} = \mathbb{X}(A_1,\dots,A_N; D_1,\dots,D_N) = \mathbb{X}((A_j,D_j)_{j=1}^N) = \bigcup_{j=1}^N (A'_j \times D_j \times A''_j),$$

where

$$A'_j := A_1 \times \ldots \times A_{j-1},\ j = 2,\dots,N,$$
$$A''_j := A_{j+1} \times \ldots \times A_N,\ j = 1,\dots,N-1,$$
$$A'_1 \times D_1 \times A''_1 := D_1 \times A''_1, \quad A'_N \times D_N \times A''_N := A'_N \times D_N.$$

For $\Sigma_j \subset A'_j \times A''_j$, $j = 1,\dots,N$ put

$$\mathcal{X}_j := \{(a'_j, z_j, a''_j) \in A'_j \times D_j \times A''_j :\ (a'_j, a''_j) \notin \Sigma_j\},$$

[(1)]That is, for all $a \in \widehat{M}$ and U_a-open neighborhood of a there exists an $\widehat{f} \in \{\widehat{f} :\ f \in \mathcal{F}\}$ such that $\widehat{f}$ does not extend holomorphically to U_a. For more details see [JarPfl 2000], Chapter 3.

where

$$\begin{aligned}
&a'_j := (a_1, \dots, a_{j-1}),\ j = 2, \dots, N,\\
&a''_j := (a_{j+1}, \dots, a_N),\ j = 1, \dots, N-1,\\
&(a'_1, z_1, a''_1) := (z_1, a''_1), \quad (a'_N, z_N, a''_N) := (a'_N, z_N).
\end{aligned}$$

We call $\mathbb{T}_{N,1}((A_j, D_j, \Sigma_j)_{j=1}^N) = \bigcup_{j=1}^N \mathcal{X}_j$ a *generalized N-fold cross* $\mathbf{T}$.

For $(a'_j, a''_j) \in A'_j \times A''_j$, $j = 1, \dots, N$, define the fiber

$$M_{(a'_j,\cdot,a''_j)} := \{z \in D_j :\ (a'_j, z, a''_j) \in M\}.$$

Our proof of Main Theorem will be based on more technically complicated result (i.e. Theorem 3.6), being an analogue of the following theorem. Moreover, we will use this result as the first inductive step in the proof of mentioned Theorem 3.6.

Theorem 2.2 (see [JarPfl 2007], Theorem 1.1)**.** *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ be locally pluriregular and let $\Sigma_j \subset A'_j \times A''_j$ be pluripolar, $j = 1, \dots, N$. Put*

$$\mathbf{X} := \mathbb{X}((A_j, D_j)_{j=1}^N), \quad \mathbf{T} := \mathbb{T}((A_j, D_j, \Sigma_j)_{j=1}^N).$$

Let $\mathcal{F} \subset \{f :\ f : c(\mathbf{T}) \setminus M \to \mathbb{C}\}$ and let $M \subset \mathbf{T}$ be such that:

- *for any $j \in \{1, \dots, N\}$ and any $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ the fiber $M_{(a'_j,\cdot,a''_j)}$ is pluripolar,*
- *for any $j \in \{1, \dots, N\}$ and any $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ there exists a closed pluripolar set $\widetilde{M}_{a,j} \subset D_j$ such that $\widetilde{M}_{a,j} \cap A_j \subset M_{(a'_j,\cdot,a''_j)}$,*
- *for any $a \in c(\mathbf{T}) \setminus M$ there exists an $r > 0$ such that for all $f \in \mathcal{F}$ there exists an $f_a \in \mathcal{O}(\mathbb{P}(a, r))$ with $f_a = f$ on $\mathbb{P}(a, r) \cap (c(\mathbf{T}) \setminus M)$*[2]*,*
- *for any $f \in \mathcal{F}$, any $j \in \{1, \dots, N\}$, and any $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ there exists a function $\widetilde{f}_{a,j} \in \mathcal{O}(D_j \setminus \widetilde{M}_{a,j})$ such that $\widetilde{f}_{a,j} = f(a'_j, \cdot, a''_j)$ on $A_j \setminus M_{a,j}$.*

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}$ such that:

- *$\widehat{M} \cap c(\mathbf{T}) \subset M$,*
- *for any $f \in \mathcal{F}$ there exists a function $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $c(\mathbf{T}) \setminus M$,*
- *$\widehat{M}$ is singular with respect to $\{\widehat{f} :\ f \in \mathcal{F}\}$,*
- *if for all $j \in \{1, \dots, N\}$ and all $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ we have $\widetilde{M}_{a,j} = \varnothing$, then $\widehat{M} = \varnothing$,*
- *if for all $j \in \{1, \dots, N\}$ and all $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ the set $\widetilde{M}_{a,j}$ is thin in D_j, then $\widehat{M}$ is analytic in $\widehat{\mathbf{X}}$.*

3. (N,k)-CROSSES

3.1. **Basic properties of (N,k)-crosses.** The following properties will be implicitly used throughout the paper.

Lemma 3.1 (Properties of (N, k)-crosses, see [JarPfl 2010], Remark 5)**.**

(i) $\mathbf{X}_{N,1} = \mathbb{X}((A_j, D_j)_{j=1}^N)$, $\widehat{\mathbf{X}}_{N,1} = \widehat{\mathbb{X}}((A_j, D_j)_{j=1}^N)$,
(ii) $\mathbf{X}_{N,k}$ *is arcwise connected,*
(iii) $\widehat{\mathbf{X}}_{N,k}$ *is connected,*
(iv) *if $D_1, \dots, D_N$ are Riemann domains of holomorphy, then $\widehat{\mathbf{X}}_{N,k}$ is a Riemann domain of holomorphy,*

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[2] $\mathbb{P}(a, r)$ denotes a polydisc in Riemann domain $D_1 \times \dots \times D_N$ centered at a with radius r. For more details see [JarPfl 2000], Chapter 1.

(v) $\mathbf{X}_{N,k} \subset \mathbf{X}_{N,k+1}$, $\widehat{\mathbf{X}}_{N,k} \subset \widehat{\mathbf{X}}_{N,k+1}$, $k = 1, \dots, N-1$,
(vi) $\mathbf{X}_{N,k} = \mathbb{X}(\mathbf{X}_{N-1,k-1}, A_N; \mathbf{X}_{N-1,k}, D_N)$, $k = 2, \dots, N-1$, $N > 2$.

The following technical lemmas will also be useful.

Lemma 3.2 ([JarPfl 2010], Lemma 4)**.** *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$ and $A_j \subset D_j$ be locally pluriregular, $j = 1, \dots, N$. Then for all $z = (z_1, \dots, z_N) \in \widehat{\mathbf{X}}_{N,k}$ we have:*

$$\mathbf{h}_{\widehat{\mathbf{X}}_{N,k-1}, \widehat{\mathbf{X}}_{N,k}}(z) = \max\big\{0, \sum_{j=1}^{N} \mathbf{h}_{A_j, D_j}(z_j) - k + 1\big\}.$$

Lemma 3.3. *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$ and $A_j \subset D_j$ be locally pluriregular, $j = 1, \dots, N$. Then for $z \in \widehat{\mathbf{X}}_{N,k}$*

$$\mathbf{h}_{\mathbf{X}_{N,k-1}, \widehat{\mathbf{X}}_{N,k}}(z) = \mathbf{h}_{\widehat{\mathbf{X}}_{N,k-1}, \widehat{\mathbf{X}}_{N,k}}(z).$$

Proof. The inequality "$\geq$" follows from properties of relative extremal function (see [JarPfl 2011], Proposition 3.2.2). To show the opposite inequality fix a $u \in \mathcal{PSH}(\widehat{\mathbf{X}}_{N,k})$ such that $u \leq 1$ and $u|_{\mathbf{X}_{N,k-1}} = 0$. Then $u|_{\widehat{\mathbf{X}}_{N,k-1}} \in \mathcal{PSH}(\widehat{\mathbf{X}}_{N,k-1})$ and $u|_{\widehat{\mathbf{X}}_{N,k-1}} \leq \mathbf{h}_{\mathbf{X}_{N,k-1}, \widehat{\mathbf{X}}_{N,k-1}}$. Using analogous(3) reasoning as in Proposition 5.1.8 (i) from [JarPfl 2011] we show that $\mathbf{h}_{\mathbf{X}_{N,k-1}, \widehat{\mathbf{X}}_{N,k-1}} \equiv 0$ on $\widehat{\mathbf{X}}_{N,k-1}$, what finishes the proof. □

3.2. **Cross theorems for (*N*,*k*)-crosses.** In this section we present the latest results considering (N,k)-crosses which will be used in the proof of the Main Theorem. Observe that our main result generalizes both of them.

Theorem 3.4 (Cross theorem for (N,k)-crosses, cf. [JarPfl 2011], Theorem 7.2.7)**.** *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$ and $A_j \subset D_j$ be locally pluriregular, $j = 1, \dots, N$. For $k \in \{1, \dots, N\}$ let $\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N)$. Then for every $f \in \mathcal{O}_{\mathcal{S}}(\mathbf{X}_{N,k})$ there exists a unique function $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k})$ such that $\widehat{f} = f$ on $\mathbf{X}_{N,k}$.*

The following result is a special case of Theorem 2.12 from [Lew 2012] being a cross theorem without singularities for generalized (N,k)-crosses.

Theorem 3.5 (Cross theorem for generalized (N,k)-crosses)**.** *Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ be pluriregular, $j = 1, \dots, N$. For $\alpha \in \mathcal{T}_k^N$ let Σ_α be a subset of A_0^α. Then for every $f \in \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k})$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k})$ such that $\widehat{f} = f$ on $\mathbf{T}_{N,k}$.*

3.3. **Extension theorem for generalized (*N*,*k*)-crosses with pluripolar singularities.** Now we state an already mentioned main technical result, being an analogue of Theorem 2.2 which is crucial for the proof of the Main Theorem. Its proof will be presented in Section 4.

Theorem 3.6. *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ be locally pluriregular, $j = 1, \dots, N$. For $\alpha \in \mathcal{Y}_k^N$ let Σ_α be a pluripolar subset of A_0^α. Let $\mathbf{W}_{N,k} \in \{\mathbf{X}_{N,k}, \mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}$, $M \subset c(\mathbf{W}_{N,k})$ and $\mathcal{F} \subset \{f : f : c(\mathbf{W}_{N,k}) \setminus M \to \mathbb{C}\}$ be such that:*

(T1) *M is pluripolar,*(4)

(3)Instead of classical cross theorem for N-fold crosses we use cross theorem for (N,k)-crosses - see Theorem 3.4.

(4)Actually we can assume less: M is such that for all $j \in \{1, \dots, N\}$ the set $\{a_j \in A_j : M_{(\cdot, a_j, \cdot)}$ is not pluripolar$\}$ is pluripolar.

(T2) *for any* $\alpha \in \mathcal{Y}_k^N$ *and any* $a \in A_0^\alpha \setminus \Sigma_\alpha$ *the fiber* $M_{a,\alpha}$ *is pluripolar,*
(T3) *for any* $\alpha \in \mathcal{Y}_k^N$ *and any* $a \in A_0^\alpha \setminus \Sigma_\alpha$ *there exists a closed pluripolar set* $\widetilde{M}_{a,\alpha} \subset D_1^\alpha$ *such that* $\widetilde{M}_{a,\alpha} \cap A_1^\alpha \subset M_{a,\alpha}$*,*[(5)]
(T4) *for any* $a \in c(\mathbf{W}_{N,k}) \setminus M$ *there exists an* $r > 0$ *such that for all* $f \in \mathcal{F}$ *there exists an* $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ *with* $f_a = f$ *on* $\mathbb{P}(a,r) \cap (c(\mathbf{W}_{N,k}) \setminus M)$*,*
(T5) *for any* $f \in \mathcal{F}$*, any* $\alpha \in \mathcal{Y}_k^N$*, and any* $a \in A_0^\alpha \setminus \Sigma_\alpha$ *there exists an* $\widetilde{f}_{a,\alpha} \in \mathcal{O}(D_1^\alpha \setminus \widetilde{M}_{a,\alpha})$ *such that* $\widetilde{f}_{a,\alpha} = f_{a,\alpha}$ *on* $A_1^\alpha \setminus M_{a,\alpha}$[(6)].

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}_{N,k}$ *such that:*

- $\widehat{M} \cap c(\mathbf{W}_{N,k}) \subset M$*,*
- *for any* $f \in \mathcal{F}$ *there exists an* $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{M})$ *such that* $\widehat{f} = f$ *on* $c(\mathbf{W}_{N,k}) \setminus M$*,*
- $\widehat{M}$ *is singular with respect to* $\{\widehat{f} : f \in \mathcal{F}\}$*,*
- *if for all* $\alpha \in \mathcal{Y}_k^N$ *and all* $a \in A_0^\alpha \setminus \Sigma_\alpha$ *we have* $\widetilde{M}_{a,\alpha} = \varnothing$*, then* $\widehat{M} = \varnothing$*,*
- *if for all* $\alpha \in \mathcal{Y}_k^N$ *and all* $a \in A_0^\alpha \setminus \Sigma_\alpha$ *the set* $\widetilde{M}_{a,\alpha}$ *is thin in* D_1^α*, then* $\widehat{M}$ *is analytic in* $\widehat{\mathbf{X}}_{N,k}$*.*

Theorem 3.6 has one immediate and useful consequence, which might be called main extension theorem for generalized (N,k)-crosses with pluripolar singularities (see analogical theorem for N-fold crosses, i.e. Theorem 10.2.6 from [JarPfl 2011]).

Proposition 3.7. *Let* D_j*,* A_j *and* Σ_α *be as in Theorem 3.6. Let*

$$\mathbf{W}_{N,k} \in \{\mathbf{X}_{N,k}, \mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}.$$

Let $M \subset \mathbf{W}_{N,k}$ *and* $\mathcal{F} \subset \mathcal{O}_{\mathcal{S}}(\mathbf{W}_{N,k} \setminus M)$ *be such that:*
(P1) $M \cap c(\mathbf{W}_{N,k})$ *is pluripolar,*
(P2) *for any* $\alpha \in \mathcal{Y}_k^N$ *and any* $a \in A_0^\alpha \setminus \Sigma_\alpha$ *the fiber* $M_{a,\alpha}$ *is pluripolar and relatively closed in* D_1^α*,*
(P3) *for any* $a \in c(\mathbf{W}_{N,k}) \setminus M$ *there exists an* $r > 0$ *such that for all* $f \in \mathcal{F}$ *there exists an* $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ *with* $f_a = f$ *on* $\mathbb{P}(a,r) \cap (c(\mathbf{W}_{N,k}) \setminus M)$*.*

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}_{N,k}$ *such that:*

- $\widehat{M} \cap c(\mathbf{W}_{N,k}) \subset M$*,*
- *for any* $f \in \mathcal{F}$ *there exists an* $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{M})$ *such that* $\widehat{f} = f$ *on* $c(\mathbf{W}_{N,k}) \setminus M$*,*
- $\widehat{M}$ *is singular with respect to* $\{\widehat{f} : f \in \mathcal{F}\}$*,*
- *if for all* $\alpha \in \mathcal{Y}_k^N$ *and all* $a \in A_0^\alpha \setminus \Sigma_\alpha$ *we have* $M_{a,\alpha} = \varnothing$*, then* $\widehat{M} = \varnothing$*,*
- *if for all* $\alpha \in \mathcal{Y}_k^N$ *and all* $a \in A_0^\alpha \setminus \Sigma_\alpha$ *the fiber* $M_{a,\alpha}$ *is thin in* D_1^α*, then* $\widehat{M}$ *is analytic in* $\widehat{\mathbf{X}}_{N,k}$*.*

Proof. Define a set $M' := M \cap c(\mathbf{W}_{N,k})$ and a family $\mathcal{F} := \{f|_{c(\mathbf{W}_{N,k})\setminus M} : f \in \mathcal{F}\}$. We show that they satisfy the assumptions of Theorem 3.6.

Indeed, for an $\alpha \in \mathcal{Y}_k^N$ and for an $a \in A_0^\alpha \setminus \Sigma_a$ define $\widetilde{M}_{a,\alpha} := M_{a,\alpha}$ and $\widetilde{f}_{a,\alpha} := f_{a,\alpha}$. Then:

- M' is pluripolar and for all $\alpha \in \mathcal{Y}_k^N$ and all $a \in A_0^\alpha \setminus \Sigma_a$ the fibers $M'_{a,\alpha}$ are pluripolar from (P1),
- for all $\alpha \in \mathcal{Y}_k^N$ and all $a \in A_0^\alpha \setminus \Sigma_a$, the set $\widetilde{M}_{a,\alpha}$ is relatively closed and pluripolar,

---

[(5)]When $k = N$ we assume that there exists an $\widetilde{M} \subset D_1 \times \ldots \times D_N$ closed pluripolar such that $\widetilde{M} \cap c(\mathbf{W}_{N,k}) \subset M$.

[(6)]When $k = N$ we assume that there exists an $\widetilde{f} \in \mathcal{O}(D_1 \times \ldots \times D_N \setminus \widetilde{M})$ such that $\widetilde{f} = f$ on $c(\mathbf{W}_{N,k}) \setminus M$.

- for all $f \in \mathcal{F}$, for all $\alpha \in \mathcal{Y}_k^N$, and all $a \in A_0^\alpha \setminus \Sigma_a$, the function $\widetilde{f}_{a,\alpha}$ is holomorphic on $D_1^\alpha \setminus \widetilde{M}_{a,\alpha}$ (cf. (P2), Definitions 1.6, 1.7 and Remark 1.8),
- from (P3) for any $a \in c(\mathbf{W}_{N,k}) \setminus M$ there exists an $r > 0$ such that for all $f \in \mathcal{F}$ there exists an $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ with $f_a = f$ on $\mathbb{P}(a,r) \cap (c(\mathbf{W}_{N,k}) \setminus M)$.

Thus from Theorem 3.6 we get the conclusion. □

As we have already mentioned in Section 2, Theorem 3.6 or, to be more precise, Proposition 3.7 implies Main Theorem. The idea of proof is based on Lemmas 10.2.5, 10.2.7, and 10.2.8 from [JarPfl 2011].

Proof that Proposition 3.7 implies Main Theorem. Let D_j, A_j, Σ_α, $\mathbf{X}_{N,k}$, $\mathbf{T}_{N,k}$, M, and $\mathcal{F}$ be as in Theorem 1.2. We have to check the assumptions of Proposition 3.7. Because M was pluripolar, for all $\alpha \in \mathcal{Y}_k^N$ there exists a pluripolar set Σ_α such that for all $a \in A_0^\alpha \setminus \Sigma_\alpha$ the fiber $M_{a,\alpha}$ is pluripolar. Moreover, because M was relatively closed, all the fibers $M_{a,\alpha}$ are relatively closed. To check the last assumption we need the following lemma.

Lemma 3.8. *Under assumptions of Theorem 1.2 for all $a \in c(\mathbf{T}_{N,k}) \setminus M$ there exists an $r > 0$ such that for any $f \in \mathcal{F}$ there exists an $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ with $f_a = f$ on $\mathbb{P}(a,r) \cap (c(\mathbf{T}_{N,k}) \setminus M)$.*

Proof. Fix an $a \in c(\mathbf{T}_{N,k}) \setminus M$. Let $\rho > 0$ be such that $\mathbb{P}(a,\rho) \cap M = \varnothing$(7). Define new crosses

$$\mathbf{X}_{N,k}^{a,\rho} := \mathbb{X}_{N,k}((A_j \cap \mathbb{P}(a_j,\rho), \mathbb{P}(a_j,\rho))_{j=1}^N)^{(8)},$$
$$\mathbf{T}_{N,k}^{a,\rho} := \mathbb{T}_{N,k}((A_j \cap \mathbb{P}(a_j,\rho), \mathbb{P}(a_j,\rho))_{j=1}^N, (\Sigma_\alpha \cap \mathbb{P}(a_\alpha,\rho))_{\alpha\in\mathcal{T}_k^N}).$$

Fix an $\alpha \in \mathcal{T}_k^N$ and an $a \in (\prod_{j:\alpha_j=0} (A_j \cap \mathbb{P}(a_j,\rho))) \setminus (\Sigma_\alpha \cap \mathbb{P}(a_\alpha,\rho))$. Then

$$(\prod_{j:\alpha_j=1} \mathbb{P}(a_j,\rho)) \setminus M_{a,\alpha} = \prod_{j:\alpha_j=1} \mathbb{P}(a_j,\rho),$$

so for any $f \in \mathcal{F}$ the function $\prod_{j:\alpha_j=1} \mathbb{P}(a_j,\rho) \ni z \mapsto f_{a,\alpha}(z)$ is holomorphic and $f \in \mathcal{O}_{\mathcal{S}}(\mathbf{T}_{N,k}^{a,\rho})$. For $\mathcal{F} = \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M)$ we additionally fix a $b \in \prod_{j:\alpha_j=1} \mathbb{P}(a_j,\rho)$. We have

$$(\prod_{j:\alpha_j=1} \mathbb{P}(a_j,\rho)) \setminus ((\Sigma_\alpha \cap \mathbb{P}(a_\alpha,\rho)) \cup M_{b,\alpha}) = (\prod_{j:\alpha_j=1} \mathbb{P}(a_j,\rho)) \setminus (\Sigma_\alpha \cap \mathbb{P}(a_\alpha,\rho))$$

and for any $f \in \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M)$ the function $(\prod_{j:\alpha_j=1} \mathbb{P}(a_j,\rho)) \setminus (\Sigma_\alpha \cap \mathbb{P}(a_\alpha,\rho)) \ni z \mapsto f_{b,\alpha}(z)$ is continuous. Thus $\mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M) \subset \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k}^{a,\rho})$. Using Theorem 3.4 for $\mathcal{F} = \mathcal{O}_{\mathcal{S}}(\mathbf{X}_{N,k} \setminus M)$ and Theorem 3.5 for $\mathcal{F} = \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M)$, we get

$$\forall\ f \in \mathcal{F}\ \exists\ \widehat{f}_a \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k}^{a,\rho}) :\ \widehat{f}_a = f \text{ on } \mathbf{T}_{N,k}^{a,\rho}\,^{(9)}.$$

Choosing $r \in (0,\rho)$ small enough to have $\mathbb{P}(a,r) \subset \widehat{\mathbf{X}}_{N,k}^{a,\rho}$ finishes the proof. □

Now, it is clear that all necessary assumptions are satisfied and we can apply Proposition 3.7. We obtain a pluripolar relatively closed set $\widehat{M}$ such that for all $f \in \mathcal{F}$ there exists an $\widehat{f}$ with $\widehat{f} = f$ on $c(\mathbf{T}_{N,k}) \setminus M$ and $\widehat{M}$ is singular with respect to $\{\widehat{f} :\ f \in \mathcal{F}\}$.

(7) Recall that M is relatively closed.

(8) From the definition of polydisc in a Riemann domain we obviously have $\mathbb{P}(a_j,\rho) \subset D_j$, $j = 1,\dots,N$.

(9) Recall that if for all $\alpha \in \mathcal{T}_k^N$ we have $\Sigma_\alpha = \varnothing$, then $\mathbf{T}_{N,k} = \mathbf{X}_{N,k}$ and $\mathbf{T}_{N,k}^{a,\rho} = \mathbf{X}_{N,k}^{a,\rho}$.

Fix an $\alpha \in \mathcal{T}_k^N$ and define $D_\alpha := D_0^\alpha$ and $G_\alpha := D_1^\alpha$. Then both D_α and G_α are Riemann domains and $\widehat{\mathbf{X}}_{N,k} \subset \overleftrightarrow{D_\alpha \times G_\alpha}$ is a Riemann domain of holomorphy. From Proposition 9.1.4 from [JarPfl 2011] there exists a pluripolar set $\Sigma'_\alpha \subset A_0^\alpha$ such that $\Sigma_\alpha \subset \Sigma'_\alpha$ and for all $a \in A_0^\alpha \setminus \Sigma'_\alpha$ the fiber $\widehat{M}_{a,\alpha}$ is singular with respect to the family $\{\widehat{f}_{a,\alpha} : f \in \mathcal{F}\}$. In particular, because every $\widehat{f}_{a,\alpha}$ is holomorphic on $(\widehat{\mathbf{X}}_{N,k})_{a,\alpha} \setminus \widehat{M}_{a,\alpha}$, we have $\widehat{M}_{a,\alpha} \subset M_{a,\alpha}$ for $a \in A_0^\alpha \setminus \Sigma'_\alpha$. Hence

$$\widehat{M} \cap \mathbf{T}'_{N,k} = \bigcup_{\alpha \in \mathcal{T}_k^N} \{z \in \widehat{M} \cap \mathcal{X}_\alpha : z_\alpha \notin \Sigma'_\alpha\} \subset M.$$

Now for every $\alpha \in \mathcal{T}_k^N$ and every $a \in A_0^\alpha \setminus \Sigma'_\alpha$ the functions $\widehat{f}_{a,\alpha}$ and $f_{a,\alpha}$ are holomorphic on the domain $D_1^\alpha \setminus M_{a,\alpha}$ (thanks to inclusion $\widehat{M} \cap \mathbf{T}'_{N,k} \subset M$) and equal on $A_1^\alpha \setminus M_{a,\alpha}$, which is not pluripolar. Thus we have an equality $\widehat{f}_{a,\alpha} = f_{a,\alpha}$ everywhere on $D_1^\alpha \setminus M_{a,\alpha}$. Because this equality holds for every α and a, we finally get $\widehat{f} = f$ on $\mathbf{T}'_{N,k} \setminus M$. □

4. Proof of Theorem 3.6

First we show that it is sufficient to prove Theorem 3.6 with $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$.

Lemma 4.1. *Theorem 3.6 with* $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$ *implies Theorem 3.6 with*

$$\mathbf{W}_{N,k} \in \{\mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}.$$

Proof. Let D_j, A_j, Σ_α, $\mathbf{X}_{N,k}$, $\mathbf{T}_{N,k}$, $\mathbf{Y}_{N,k}$, $M \subset c(\mathbf{W}_{N,k})$ and $\mathcal{F} \subset \{f : f : \mathbf{W}_{N,k} \setminus M \to \mathbb{C}\}$, where $\mathbf{W}_{N,k} \in \{\mathbf{T}_{N,k}.\mathbf{Y}_{N,k}\}$, be like in Theorem 3.6. Assume that this theorem is true with $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$.

Observe that $c(\mathbf{Y}_{N,k}) = c(\mathbf{X}_{N,k}) \setminus \Delta$ and $c(\mathbf{T}_{N,k}) = c(\mathbf{X}_{N,k}) \setminus \widetilde{\Delta}$, where

$$\Delta := \bigcap_{\alpha \in \mathcal{T}_k^N} \{a \in A_1 \times \ldots \times A_N : a_\alpha \in \Sigma_\alpha\},$$

$$\widetilde{\Delta} := \bigcap_{\alpha \in \mathcal{Y}_k^N} \{a \in A_1 \times \ldots \times A_N : a_\alpha \in \Sigma_\alpha\},$$

are pluripolar subsets of $c(\mathbf{X}_{N,k})$, where a_α denotes the projection of a on A_0^α.

Define $M' := M \cup \widetilde{\Delta} \subset c(\mathbf{X}_{N,k})$. Then $c(\mathbf{X}_{N,k}) \setminus \widetilde{\Delta}) \setminus M = c(\mathbf{X}_{N,k}) \setminus M'$ and

$$(*) \qquad c(\mathbf{T}_{N,k}) \setminus M = (c(\mathbf{X}_{N,k}) \setminus \Delta) \setminus M \subset (c(\mathbf{X}_{N,k} \setminus \widetilde{\Delta}) \setminus M \text{ for } M \subset c(\mathbf{T}_{N,k}),$$

$$(**) \qquad c(\mathbf{Y}_{N,k}) \setminus M = (c(\mathbf{X}_{N,k}) \setminus \widetilde{\Delta}) \setminus M \text{ for } M \subset c(\mathbf{Y}_{N,k}).$$

Define $\mathcal{F}' := \{f|_{c(\mathbf{X}_{N,k}) \setminus M'} : f \in \mathcal{F}\}$. Then M' is pluripolar and for all $\alpha \in \mathcal{Y}_k^N$ and all $a \in A_0^\alpha \setminus \Sigma_\alpha$ we have $\widetilde{\Delta}_{a,\alpha} = \varnothing$, so $M'_{a,\alpha} = M_{a,\alpha}$. Thus M' and the family $\mathcal{F}'$ satisfies assumptions of Theorem 3.6 with $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$. Then there exists $\widehat{M'} \subset \widehat{\mathbf{X}}_{N,k}$, relatively closed, pluripolar, having all properties from the thesis. Properties $(*)$ and $(**)$ give us the conclusion for $\mathbf{W}_{N,k} \in \{\mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}$. □

Proof of Theorem 3.6 with $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$.

Step 1. Theorem 3.6 is true for any N when $k = 1$ (Theorem 2.2) and when $k = N$ (in this case we assumed the thesis).

Step 2. In particular, theorem is true for $N = 2$, $k = 1, 2$. Assume we already have Theorem 3.6 for $(N-1, k)$, where $k \in \{1, \ldots, N-1\}$, and for $(N,1)$,…,$(N, k-1)$, where $k \in \{2, \ldots, N-1\}$. We need to prove it for (N, k).

Step 3. Fix an $s \in \{1, \ldots, N\}$ (to simplify the notation let $s = N$). Let

$$Q_N := \{a_N \in A_N : M_{(\cdot, a_N)} \text{ is not pluripolar}\}.$$

Then Q_N is pluripolar. Define

$$\mathbf{X}^{(s)}_{N-1,k} := \mathbb{X}_{N-1,k}((A_j, D_j)_{j=1, j\neq s}^N),\ s = 1, \dots, N,$$

in particular

$$\mathbf{X}^{(N)}_{N-1,k} = \mathbf{X}_{N-1,k} := \mathbb{X}_{N-1,k}((A_j, D_j)_{j=1}^{N-1}).$$

Fix an $a_N \in A_N \setminus Q_N$ and define a family $\{f(\cdot, a_N) : \ f \in \mathcal{F}\} \subset \{f : \ f : c(\mathbf{X}_{N-1,k}) \to \mathbb{C}\}$. Then:

- $M_{(\cdot,a_N)} \subset c(\mathbf{X}_{N-1,k})$ is pluripolar.
- For any $\alpha' \in \mathcal{Y}_k^{N-1}$ and any $a' \in A_0^{\alpha'} \setminus \Sigma_{\alpha'}$[(10)] the fiber $(M_{(\cdot,a_N)})_{a',\alpha'}$ equals $M_{a,\alpha}$, where $a = (a', a_N)$ and $\alpha = (\alpha', 0)$, so it is pluripolar.
- For $\alpha' \in \mathcal{Y}_k^{N-1}$, $a' \in A_0^{\alpha'} \setminus \Sigma_{\alpha'}$ we define $\widetilde{M}_{a',\alpha'} := \widetilde{M}_{a,\alpha}$, where $a = (a', a_N)$, $\alpha = (\alpha', 0)$. Then $\widetilde{M}_{a',\alpha'} \subset D_1^{\alpha} = D_1^{\alpha'}$ is closed, pluripolar and $\widetilde{M}_{a',\alpha'} \cap A_1^{\alpha'} \subset M_{a',\alpha'}$.
- For any $a' \in c(\mathbf{X}_{N-1,k}) \setminus M_{(\cdot,a_N)}$ there exists an $r > 0$ (the same as for $a = (a', a_N)$) such that for any $f \in \mathcal{F}$ there exists $f_{a'} \in \mathcal{O}(\mathbb{P}(a', r))$ such that $f_{a'} = f(\cdot, a_N)$ on $\mathbb{P}(a', r) \cap (c(\mathbf{X}_{N-1,k}) \setminus M_{(\cdot,a_N)})$.
- For $f \in \mathcal{F}$, for any $\alpha' \in \mathcal{Y}_k^{N-1}$ and any $a' \in A_0^{\alpha'} \setminus \Sigma_{\alpha'}$, define $\widetilde{f}_{a',\alpha'} := \widetilde{f}_{a,\alpha} \in \mathcal{O}(D_1^{\alpha} \setminus \widetilde{M}_{a,\alpha}) = \mathcal{O}(D_1^{\alpha'} \setminus \widetilde{M}_{a',\alpha'})$, where $a = (a', a_N)$, $\alpha = (\alpha', 0)$. Then $\widetilde{f}_{a',\alpha'} = f_{a',\alpha'}$ on $A_1^{\alpha'} \setminus (M_{(\cdot,a_N)})_{a',\alpha'}$.

From the inductive assumption we get a relatively closed pluripolar set $\widehat{M}_{a_N} \subset \widehat{\mathbf{X}}_{N-1,k}$ such that:

- $\widehat{M}_{a_N} \cap c(\mathbf{X}_{N-1,k}) \subset M_{(\cdot,a_N)}$,
- for any $f \in \mathcal{F}$ there exists an $\widehat{f}_{a_N} \in \mathcal{O}(\widehat{\mathbf{X}}_{N-1,k} \setminus \widehat{M}_{a_N})$ such that $\widehat{f}_{a_N} = f(\cdot, a_N)$ on $c(\mathbf{X}_{N-1,k}) \setminus M_{(\cdot,a_N)}$,
- $\widehat{M}_{a_N}$ is singular with respect to $\{\widehat{f}_{a_N} : \ f \in \mathcal{F}\}$,
- if for all $\alpha' \in \mathcal{Y}_k^{N-1}$ and all $a' \in A_0^{\alpha'} \setminus \Sigma_{\alpha'}$, we have $\widetilde{M}_{a',\alpha'} = \varnothing$, then $\widehat{M}_{a_N} = \varnothing$,
- if for all $\alpha' \in \mathcal{Y}_k^{N-1}$ and all $a' \in A_0^{\alpha'} \setminus \Sigma_{\alpha'}$, the set $\widetilde{M}_{a',\alpha'}$ is thin in $D_1^{\alpha'}$, then $\widehat{M}_{a_N}$ is analytic in $\widehat{\mathbf{X}}_{N-1,k}$.

Define a new cross

$$\mathbf{Z}_N := \mathbb{X}(c(\mathbf{X}_{N-1,k}), A_N; \widehat{\mathbf{X}}_{N-1,k}, D_N).$$

Observe that $\mathbf{Z}_N$ with original M, $\Sigma_{(0,1)} := \Sigma_{(0,\dots,0,1)}$, $\Sigma_{(1,0)} := Q_N$, and the family $\mathcal{F}$ satisfies all the assumptions of Theorem 3.6 with $N = 2$, $k = 1$. Indeed:

- For all $a' \in c(\mathbf{X}_{N-1,k}) \setminus \Sigma_{(0,\dots,0,1)}$ and all $a_N \in A_N \setminus Q_N$ the fibers $M_{(a',\cdot)}$, $M_{(\cdot,a_N)}$ are pluripolar from (T1), (T2) and definition of Q_N.
- For all $a' \in c(\mathbf{X}_{N-1,k}) \setminus \Sigma_{(0,\dots,0,1)}$ from (T3) there exists an $\widetilde{M}_{a'} \subset D_N$ closed pluripolar such that $\widetilde{M}_{a'} \cap A_N \subset M_{(a',\cdot)}$. For $a_N \in A_N \setminus Q_N$ set $\widetilde{M}_{a_N} := \widehat{M}_{a_N}$. Then $\widetilde{M}_{a_N}$ is closed pluripolar in $\widehat{\mathbf{X}}_{N-1,k}$ and $\widetilde{M}_{a_N} \cap c(\mathbf{X}_{N-1,k}) \subset M_{(\cdot,a_N)}$.
- For all $(a', a_N) \in (c(\mathbf{X}_{N-1,k}) \times A_N) \setminus M$ from (T4) there exists an $r > 0$ such that for all $f \in \mathcal{F}$ there exists an $f_{(a',a_N)} \in \mathcal{O}(\mathbb{P}((a', a_N), r))$ such that

$$f_{(a',a_N)} = f \text{ on } \mathbb{P}((a', a_N), r) \cap (c(\mathbf{X}_{N,k} \setminus M)).$$

[(10)]By $A_0^{\alpha'}$ we denote the product $\prod_{j\in\{1,\dots,N-1\}:\, \alpha'_j=0} A_j$. Analogously for $A_1^{\alpha'}$ and $D_1^{\alpha'}$.

- For all $a' \in c(\mathbf{X}_{N-1,k}) \setminus \Sigma_{\alpha=(0,\dots,0,1)}$ from (T5) there exists an $f_{a'} \in \mathcal{O}(D_N \setminus \widetilde{M}_{a'})$ such that $f_{a'} = f$ on $A_N \setminus M_{(a',\cdot)}$. For an $a_N \in A_N \setminus Q_N$ define $f_{a_N} := \widehat{f}_{a_N}$. Then $f_{a_N} \in \mathcal{O}(\widehat{\mathbf{X}}_{N-1,k} \setminus \widetilde{M}_{a_N})$ and $f_{a_N} = f$ on $c(\mathbf{X}_{N-1,k}) \setminus M_{(\cdot,a_N)}$.

Then there exists an $\widehat{M}_N \subset \widehat{\mathbf{Z}}_N$ relatively closed pluripolar such that:

- $\widehat{M}_N \cap c(\mathbf{X}_{N,k}) \subset M$,
- for any $f \in \mathcal{F}$ there exists an $\widehat{f}_N \in \mathcal{O}(\widehat{\mathbf{Z}}_N \setminus \widehat{M}_N)$ such that $\widehat{f}_N = f$ on $c(\mathbf{X}_{N,k}) \setminus M$,
- $\widehat{M}_N$ is singular with respect to $\{\widehat{f}_N : f \in \mathcal{F}\}$,
- if for all $a' \in c(\mathbf{X}_{N-1,k}) \setminus \Sigma_{(0,\dots,0,1)}$ we have $\widetilde{M}_{a'} = \varnothing$ and for all $a_N \in A_N \setminus Q_N$ we have $\widetilde{M}_{a_N} = \varnothing$, then $\widehat{M}_N = \varnothing$,
- if for all $a' \in c(\mathbf{X}_{N-1,k}) \setminus \Sigma_{(0,\dots,0,1)}$ the set $\widetilde{M}_{a'}$ is thin in D_N and for all $a_N \in A_N \setminus Q_N$ the set $\widetilde{M}_{a_N}$ is thin in $\widehat{\mathbf{X}}_{N-1,k}$, then $\widehat{M}_N$ is analytic in $\widehat{\mathbf{Z}}_N$.

We repeat the reasoning above for all $s = 1,\dots,N-1$, obtaining a family of functions $\{\widehat{f}_s\}_{s=1}^N$ such that for any $s \in \{1,\dots,N\}$ we have $\widehat{f}_s = f$ on $c(\mathbf{X}_{N,k}) \setminus M$. Define a new function

$$F_f(z) := \begin{cases} \widehat{f}_1(z) & \text{for} \quad z \in \widehat{\mathbf{Z}}_1 \setminus \widehat{M}_1 \\ \vdots & \\ \widehat{f}_N(z) & \text{for} \quad z \in \widehat{\mathbf{Z}}_N \setminus \widehat{M}_N \end{cases}.$$

Assume for a moment that we have the following lemma.

Lemma 4.2. *Function F_f is well defined on* $\left(\bigcup\limits_{s=1}^N \mathbf{Z}_s\right) \setminus \left(\bigcup\limits_{s=1}^N \widehat{M}_s\right)$.

Step 4. Define a 2-fold cross

$$\mathbf{Z} := \mathbb{X}(\mathbf{X}_{N-1,k-1}, A_N; \widehat{\mathbf{X}}_{N-1,k}, D_N) \subset \bigcup_{s=1}^N \mathbf{Z}_s,$$

a pluripolar set

$$\widetilde{M} := \left(\bigcup_{s=1}^N \widehat{M}_s\right) \cap (\mathbf{X}_{N-1,k-1} \times A_N)$$

and a family

$$\widetilde{\mathcal{F}} := \{\widetilde{f} := F_f|_{(\mathbf{X}_{N-1,k-1} \times A_N) \setminus \widetilde{M}} : f \in \mathcal{F}\}.$$

We show that $\mathbf{Z}$, $\widetilde{M}$, and $\widetilde{\mathcal{F}}$ satisfy the assumptions of Theorem 3.6 with $N = 1$ and $k = 1$.

- $\widetilde{M}$ is pluripolar in $\mathbf{X}_{N-1,k-1} \times A_N$, so there exist pluripolar sets $P \subset \mathbf{X}_{N-1,k-1}$, $Q \subset A_N$ such that for all $z' \in \mathbf{X}_{N-1,k-1} \setminus P$, $a_N \in A_N \setminus Q$, the fibers $\widetilde{M}_{(z',\cdot)}$, $\widetilde{M}_{(\cdot,a_N)}$ are pluripolar.
- Let $z' \in \mathbf{X}_{N-1,k-1} \setminus P$. Then there exists an $s \in \{1,\dots,N-1\}$ such that

$$(\star) \qquad \{z'\} \times D_N \subset \mathbf{X}_{N-1,k}^{(s)} \times A_s.$$

  Indeed, let $z' \in \mathbf{X}_{N-1,k-1}$. Then $z' = z'_\alpha$ for some $\alpha \in \{0,1\}^{N-1}$, $|\alpha| = k-1$, where $z'_\alpha = (z_{\alpha_1},\dots,z_{\alpha_{N-1}})$ and $z_{\alpha_j} = a_j \in A_j$ when $\alpha_j = 0$, $z_{\alpha_j} = z_j \in D_j$ otherwise. We may assume that $z' = (z_1,\dots,z_{k-1},a_k,\dots,a_{N-1})$. Set $s = k$. Fix a $z_N \in D_N$. Then $(z_1,\dots,z_{k-1},a_k,\dots,a_{N-1},z_N) \in \{z'\} \times D_N$ and

$$(z_1,\dots,z_{k-1},a_k,\dots,a_{N-1},z_N) \in \mathbf{X}_{N-1,k}^{(s)} \times A_s.$$

Define $\widetilde{M}_{z'} := (\widehat{M}_s)_{(z',\cdot)}$. Then $\widetilde{M}_{z'}$ is pluripolar relatively closed in D_N and $\widetilde{M}_{z'} \cap A_N \subset \widetilde{M}_{(z',\cdot)}$.
For an $a_N \in A_N \setminus Q$ define $\widetilde{M}_{a_N} := (\widehat{M}_N)_{(\cdot,a_N)}$ - relatively closed pluripolar in $\widehat{\mathbf{X}}_{N-1,k}$ such that $\widetilde{M}_{a_N} \cap \mathbf{X}_{N-1,k-1} \subset \widetilde{M}_{(\cdot,a_N)}$.

- For any $(z', a_N) \in (\mathbf{X}_{N-1,k-1} \times A_N) \setminus \widetilde{M}$ there exist an $s \in \{1, \ldots, N-1\}$ and an $r > 0$ such that $\mathbb{P}((z', a_N), r) \subset \widehat{\mathbf{Z}}_s \setminus \widehat{M}_s$. Then $\widehat{f}_s \in \mathcal{O}(P((z', a_N), r))$ and $\widehat{f}_s = F_f = \widetilde{f}$ on $P((z', a_N), r) \cap ((\mathbf{X}_{N-1,k-1} \times A_N) \setminus \widetilde{M})$.
- For a $z' \in \mathbf{X}_{N-1,k-1} \setminus P$ choose an s to have $(\star)$ and define $\widetilde{f}_{z'} := \widehat{f}_s(z', \cdot)$. Then $\widetilde{f}_{z'}$ is holomorphic on $D_N \setminus \widetilde{M}_{z'}$ and equals $\widetilde{f}(z', \cdot)$ on $A_N \setminus \widetilde{M}_{(z',\cdot)}$.
For an $a_N \in A_N \setminus Q$ define $\widetilde{f}_{a_N} := \widehat{f}_s(\cdot, a_N)$. Then $\widetilde{f}_{a_N}$ is holomorphic on $\widehat{\mathbf{X}}_{N-1,k} \setminus \widetilde{M}_{a_N}$ and equals $\widetilde{f}(\cdot, a_N)$ on $\mathbf{X}_{N-1,k-1} \setminus \widetilde{M}_{(\cdot,a_N)}$.

Now from Theorem 3.6 there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\mathbf{Z}}$ such that:

- $\widehat{M} \cap (\mathbf{X}_{N-1,k-1} \times A_N) \subset \widetilde{M}$, in particular, $\widehat{M} \cap c(\mathbf{X}_{N,k}) \subset M$,
- for any $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{Z}} \setminus \widehat{M})$ such that $\widehat{f} = \widetilde{f}$ on $(\mathbf{X}_{N-1,k-1} \times A_N) \setminus \widetilde{M}$, in particular $\widehat{f} = f$ on $c(\mathbf{X}_{N,k}) \setminus M$,
- $\widehat{M}$ is singular with respect to $\{\widehat{f} :\ f \in \mathcal{F}\}$,
- if for all $z' \in \mathbf{X}_{N-1,k-1} \setminus P$ we have $\widetilde{M}_{z'} = \varnothing$ and for all $a_N \in A_N \setminus Q$ $\widetilde{M}_{a_N} = \varnothing$, then $\widehat{M} = \varnothing$,
- if for all $z' \in \mathbf{X}_{N-1,k-1} \setminus P$ the set $\widetilde{M}_{z'}$ is thin in D_N and for all $a_N \in A_N \setminus Q$ the set $\widetilde{M}_{a_N}$ is thin in $\widehat{\mathbf{X}}_{N-1,k}$, then $\widehat{M}$ is analytic in $\widehat{\mathbf{Z}}$.

Now assume that for any $\alpha \in \mathcal{Y}_k^N$ and any $a \in A_0^\alpha \setminus \Sigma_\alpha$ we have $\widetilde{M}_{a,\alpha} = \varnothing$. Then for any $s \in \{1, \ldots, N\}$ and for any $a_s \in A_s \setminus Q_s$ we have $\widehat{M}_{a_s} = \varnothing$ what implies that for all $s \in \{1, \ldots, N\}$ we have $\widehat{M}_s = \varnothing$. Then from the definitions of $\widetilde{M}_{z'}$ and $\widetilde{M}_{a_N}$ we get that for any $z' \in \mathbf{X}_{N-1,k-1} \setminus P$ we have $\widetilde{M}_{z'} = \varnothing$ and for all $a_N \in A_N \setminus Q$ we have $\widetilde{M}_{a_N} = \varnothing$, thus $\widehat{M} = \varnothing$.

Analogously if for all $\alpha \in \mathcal{Y}_k^N$ and all $a \in A_0^\alpha \setminus \Sigma_\alpha$ the fiber $\widetilde{M}_{a,\alpha}$ is thin in D_1^α, then for any $s \in \{1, \ldots, N\}$ and any $a_s \in A_s \setminus Q_s$ the set $\widehat{M}_{a_s}$ is analytic (thus thin) in $\widehat{\mathbf{X}}_{N-1,k}^{(s)}$, so for all $s \in \{1, \ldots, N\}$ the set $\widehat{M}_s$ is analytic in $\widehat{\mathbf{Z}}_s$. Because fibers of analytic sets are also analytic we get that for any $z' \in \mathbf{X}_{N-1,k-1} \setminus P$ the set $\widetilde{M}_{z'}$ is thin in D_N and for any $a_N \in A_N \setminus Q$ the set $\widetilde{M}_{a_N}$ is thin in $\widehat{\mathbf{X}}_{N-1,k}$. Then, finally, $\widehat{M}$ is analytic in $\widehat{\mathbf{Z}}$.

Now we show that $\widehat{\mathbf{X}}_{N,k} \subset \widehat{\mathbf{Z}}$. First observe that if $z = (z', z_N) \in \widehat{\mathbf{X}}_{N,k}$, then $z' \in \widehat{\mathbf{X}}_{N-1,k}$. From Lemma 3.3 for $(z_1, \ldots, z_N) = (z', z_N) \in \widehat{\mathbf{X}}_{N,k}$ we get

$$(\ddagger) \qquad \mathbf{h}_{\mathbf{X}_{N-1,k-1},\widehat{\mathbf{X}}_{N-1,k}}(z') + \mathbf{h}_{A_N,D_N}(z_N) = \mathbf{h}_{\widehat{\mathbf{X}}_{N-1,k-1}}(z') + \mathbf{h}_{A_N,D_N}(z_N)\ .$$

For $z \in \widehat{\mathbf{X}}_{N-1,k-1} \subset \widehat{\mathbf{X}}_{N,k}$ $(\ddagger)$= $\mathbf{h}_{A_N,D_N}(z_N)$, which is less than 1 from properties of relative extremal function, and for $z \in \widehat{\mathbf{X}}_{N,k} \setminus \widehat{\mathbf{X}}_{N-1,k-1}$ we use Lemma 3.2

$$(\ddagger) = \left(\sum_{j=1}^{N-1} \mathbf{h}_{A_j,D_j}(z_j) \right) - k + 1 + \mathbf{h}_{A_N,D_N}(z_N) < k - 1 + 1 = 1\ .$$

To show the opposite inclusion take $(z_1, \dots, z_N) = (z', z_N) \in \widehat{\mathbf{Z}}$. From properties of relative extremal function and Lemma 3.2 we get

$$\left(\sum_{j=1}^{N-1} \mathbf{h}_{A_j,D_j}(z_j)\right) + \mathbf{h}_{A_N,D_N}(z_N) \le \mathbf{h}_{\widehat{\mathbf{X}}_{N-1,k-1}}(z') + k - 1 + \mathbf{h}_{A_N,D_N}(z_N)$$

$$\le \mathbf{h}_{\mathbf{X}_{N-1,k-1},\widehat{\mathbf{X}}_{N-1,k}}(z') + \mathbf{h}_{A_N,D_N}(z_N) + k - 1 < 1 + k - 1 = k\ .$$

Thus, it is left to prove Lemma 4.2:

Proof of Lemma 4.2. Fix s and p. We want to show that $\widehat{f}_s = \widehat{f}_p$ on $(\mathbf{Z}_s \cap \mathbf{Z}_p) \setminus (\widehat{M}_s \cup \widehat{M}_p)$. To simplify the notation we may assume that $s = N-1$ and $p = N$.

Step 1. Every connected component of $\mathbf{Z}_{N-1} \cap \mathbf{Z}_N$ contains part of the center.

From the definition of $\mathbf{Z}_{N-1}$ and $\mathbf{Z}_N$ we have

$$\mathbf{Z}_{N-1} \cap \mathbf{Z}_N = (A_1 \times \ldots \times A_{N-2} \times D_{N-1} \times A_N) \cup (A_1 \times \ldots A_{N-1} \times D_N)$$

$$\cup (\widehat{\mathbf{X}}_{N-2,k} \times A_{N-1} \times A_N).$$

First take $B_1 := A_1 \times \ldots \times A_{N-2} \times A_{N-1} \times D_N$. Since the product of a not connected set with any set is not connected, connected components of B_1 are products of connected components of A_j, $j = 1, \dots, N-1$, and D_N. Since the last set is connected, every connected component of B_1 "contains" D_N (in the sense of last place in the product) thus it contains a part of the center $A_1 \times \ldots \times A_N$.

Case of $B_2 := A_1 \times \ldots \times A_{N-2} \times D_{N-1} \times A_N$ is similar.

Now take $B_3 := \widehat{\mathbf{X}}_{N-2,k} \times A_{N-1} \times A_N$. As in the previous cases, since $\widehat{\mathbf{X}}_{N-2,k}$ is connected, every connected component of B_2 "contains" whole $\widehat{\mathbf{X}}_{N-2,k}$ in the product. Since $\widehat{\mathbf{X}}_{N-2,k}$ contains $A_1 \times \ldots \times A_{N-2}$, every connected component of B_2 must contain part of the center.

Step 2. One connected component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$ contains whole $\mathbf{Z}_{N-1} \cap \mathbf{Z}_N$.

Intersection $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$ contains cross $\mathbf{X}_{N,1}$ which is connected and contains the center. Thus the whole center must lay in one connected component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$. Now take any connented component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$ which intersects $\mathbf{Z}_{N-1} \cap \mathbf{Z}_N$. From Step 1 it must contain part of the center, so there is only one connected component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$ intersecting (thus containing) $\mathbf{Z}_{N-1} \cap \mathbf{Z}_N$.

Step 3. Every connected component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$ with $\widehat{M}_{N-1} \cup \widehat{M}_N$ deleted is a domain, thus it is a connected component of $(\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N) \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$.

Take any connected component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$, name it Ω. Then Ω is a domain. The set $\widehat{M}_{N-1}$ is pluripolar and relatively closed in $\widehat{\mathbf{Z}}_{N-1}$, thus it is pluripolar and relatively closed in Ω, so $\Omega \setminus \widehat{M}_{N-1}$ is still a domain. Because $\widehat{M}_N$ is relatively closed and pluripolar in $\widehat{\mathbf{Z}}_N$, it is relatively closed and pluripolar in $\Omega \setminus \widehat{M}_{N-1}$. So $\Omega \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$ is a domain.

Step 4. One connected component of $(\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N) \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$ contains whole set $(\mathbf{Z}_{N-1} \cap \mathbf{Z}_N) \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$.

It follows immediately from Step 2 and Step 3.

Step 5. $\widehat{f}_{N-1} = \widehat{f}_N$ on $(\mathbf{Z}_{N-1} \cap \mathbf{Z}_N) \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$.

Let Ω be a connected component from Step 4. Then both $\widehat{f}_{N-1}$ and $\widehat{f}_N$ are defined on Ω. On the non-pluripolar center we have $\widehat{f}_{N-1} = \widehat{f}_N$. Ω is a domain and contains the center, so $\widehat{f}_{N-1} = \widehat{f}_N$ on Ω. Moreover, Ω contains $(\mathbf{Z}_{N-1} \cap \mathbf{Z}_N) \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$, what finishes the proof. □

The proof of Theorem 3.6 is finished. □

Example 4.3. In the proof of Theorem 3.6 with $k = 1$ we do not need cross $\widehat{\mathbf{Z}}$ - it is sufficient to take $\widehat{\mathbf{Z}}_N$ (see [JarPfl 2010] for details), however in the case when $k > 1$ Step 4 is necessary. Indeed, let $A_1 = A_2 = A_3 = (-1, 1)$, $D_1 = D_2 = D_3 = \mathbb{D}$, $\mathbf{X}_{3,2} := \mathbb{X}_{3,2}((A_j, D_j)_{j=1}^3)$, $\mathbf{Z}_3 := \mathbb{X}(A_1 \times A_2, A_3; \widehat{\mathbf{X}}_{2,2}, D_3)$. Then $\widehat{\mathbf{X}}_{2,2} = D_1 \times D_2$,

$$\widehat{\mathbf{Z}}_3 := \{z \in D_1 \times D_2 \times D_3 : \ \mathbf{h}_{A_1 \times A_2, D_1 \times D_2}(z_1, z_2) + \mathbf{h}_{A_3,D_3}(z_3) < 1\} =$$
$$= \{z \in D_1 \times D_2 \times D_3 : \ \max\{h_{A_j,D_j}(z_j), \ j = 1, 2\} + \mathbf{h}_{A_3,D_3}(z_3) < 1\},$$

and $\mathbf{h}_{A_j,D_j}(\zeta) = \frac{2}{\pi} \left|\operatorname{Arg}\left(\frac{1+\zeta}{1-\zeta}\right)\right|$, $\zeta \in \mathbb{D}$, $j = 1, 2, 3$ (see Example 3.2.20 (a) in [JarPfl 2011]). Take $z = (0, w, w)$, where $w = \frac{i}{\sqrt{3}}$. Then $z \in \mathbf{X}_{3,2}$ but $z \notin \widehat{\mathbf{Z}}_3$.

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